JOURNAL OF NUMBER THEORY 24, 134–153 (1986)

On the A-Dependence of the Linear Sieve with Application to Polynomial Sequences

G. BANTLE

Department of Mathematics, Universität Ulm, D-7900 Ulm, West Germany

Communicated by H. Halberstam

Received January 2, 1984

Let $\mathscr{A}$ denote a finite sequence of integers and put $\mathscr{A}_d = \{a \in \mathscr{A} : a \equiv 0(d)\}$. Let $\mathscr{P}$ denote a set of distinct primes and write $\mathscr{P}(z) = \prod_{p<z, p \in \mathscr{P}} p$, $z \geqslant 2$. Assume that there exists a convenient approximation X to $|\mathscr{A}|$ and a nonnegative, multiplicative, arithmetic function $\omega(d)$ on the divisors d of $\mathscr{P}(z)$, such that the remainder $R_d := |\mathscr{A}_d| - (\omega(d)/d)X$ are small on average over all divisors d of $\mathscr{P}(z)$ that are less than a certain number y. Jurkat and Richert introduced the well-known functions $f(u)$ and $F(u)$ to show that under the linear sieve condition

$$\prod_{v \leqslant p < w} \left(1 - \frac{\omega(p)}{p}\right)^{-1} \leqslant \frac{\log w}{\log v}\left(1 + \frac{A}{\log v}\right), \qquad 2 \leqslant v < w, \tag{1}$$

the upper and lower sieve bounds

$$S(\mathscr{A}, \mathscr{P}, z) := \sum_{a \in \mathscr{A}, (a, \mathscr{P}(z)) = 1} 1 \lesseqgtr X \prod_{p < z, p \in \mathscr{P}} \left(1 - \frac{\omega(p)}{p}\right) \times \left\{ \begin{matrix} F(u) + \\ f(u) - \end{matrix} H \right\}^{+}_{-} \sum_{d < y, d | \mathscr{P}(z)} |R_d|, \tag{2}$$

where $u = \log y / \log z$, $y \geqslant z \geqslant 2$ and $H = O((\log X)^{-1/14})$. Iwaniec was the first who made the dependence of H on A explicit by proving (2) with $H = O(e^{\sqrt{A}} e^{-u} (\log y)^{-1/3})$. Richert improved this by

$$H = O(A^8 e^{-u} (\log z)^{-1/3}). \tag{3}$$

The main result of this paper is that (2) is valid with $H = O((A \log A)^{13/3} (\log z)^{-1/3})$, if u is bounded. Sieving polynomial sequences, the natural condition instead of (1) is the well-known condition $\Omega_2(1, A_2)$ in the notation of Halberstam and Richert about sieve methods.

A simple calculation shows, that Richerts' result (3) could be replaced by

$$H = O(e^{8A_2/\log 2} e^{-u} (\log z)^{-1/3}). \tag{4}$$

With the method of this paper, (4) can be improved by $H = O_\varepsilon(e^{\varepsilon A_2} (\log z)^{-1/3})$ in the case where u is bounded.

0022-314X/86 $3.00

In the second part of the paper, the results are applied to the sieving of polynomial sequences and two examples are given, where the error terms are explicit in the discriminant of the corresponding polynomial.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

As usual, let $\mathscr{A}$ denote a finite sequence of integers and put $\mathscr{A}_d = \{a \in \mathscr{A} : a \equiv 0(d)\}$. Let $\mathscr{P}$ denote a set of distinct primes and write $\mathscr{P}(z) = \prod_{p<z, p \in \mathscr{P}} p$, $z \geqslant 2$. Sieving the sequence $\mathscr{A}$ by a sieve means to eliminate all elements from $\mathscr{A}$ which are divisible by a prime $p \in \mathscr{P}$, $p < z$. A basic problem, therefore, is to estimate the sifting function

$$S(\mathscr{A}, \mathscr{P}, z) = \sum_{\substack{a \in \mathscr{A} \\ (a, \mathscr{P}(z)) = 1}} 1 = \sum_{d \mid \mathscr{P}(z)} \mu(d) |\mathscr{A}_d| \tag{1.1}$$

under certain conditions on $\mathscr{A}$ and $\mathscr{P}$.

We assume that there exists a convenient approximation X to $|\mathscr{A}|$ and a nonnegative, multiplicative, arithmetic function $\omega(d)$ on the divisors d of $\mathscr{P}(z)$, such that the remainder $R_d := |\mathscr{A}_d| - (\omega(d)/d) X$ are small, on average, over all divisors d on $\mathscr{P}(z)$, being less than a certain number y. For convenience we put $\omega(p) = 0$ if $p \notin \mathscr{P}$ and set $V(z) = \prod_{p<z} (1 - \omega(p)/p)$. As a natural condition we require that $\omega(p)/p < 1$.

Jurkat and Richert [7] introduced the well-known functions $f(u)$ and $F(u)$ to show under further conditions on $\omega(p)$ the upper and lower bounds

$$S(\mathscr{A}, \mathscr{P}, z) \leqslant XV(z)\{F(u) + H\} + \sum_{\substack{d < y \\ d \mid \mathscr{P}(z)}} |R_d| \tag{1.2}$$

and

$$S(\mathscr{A}, \mathscr{P}, z) \geqslant XV(z)\{f(u) - H\} - \sum_{\substack{d > y \\ d \mid \mathscr{P}(z)}} |R_d|, \tag{1.3}$$

where

$$u = \frac{\log y}{\log z}, \qquad y \geqslant z \geqslant 2 \tag{1.4}$$

and

$$H = O((\log X)^{-1/14}) \qquad \text{(cf. [3, Theorem 8.4])}.$$

These estimates are, in a certain sense, best possible, because there exist examples, for which (1.2) and (1.3) respectively hold with equality; i.e., the main terms $XV(z)\,F(u)$ and $XV(z)\,f(u)$ cannot be improved.

What could be done is, apart from replacing the sum over the remainders by bilinear forms (cf. [6]), to weaken the conditions on $\omega(p)$ and to make the dependence on H from these conditions explicit. Under the assumption that $\omega(p)$ is at most 1, on average, in the sense that there exists a positive number $A \geqslant 2$ such that

$$(\Omega_2)\colon\quad \frac{V(v)}{V(w)} = \prod_{v \leqslant p < w} \left(1 - \frac{\omega(p)}{p}\right)^{-1} \leqslant \frac{\log w}{\log v}\left(1 + \frac{A}{\log v}\right), \qquad 2 \leqslant v < w,$$

Iwaniec [5] proved that (1.2) and (1.3) hold true with

$$H = e^{\sqrt{A}}\, e^{-u \log u + u \log\log 3u + O(u)} (\log y)^{-1/3}.$$

In an unpublished manuscript Richert [11] simplified the proof of Iwaniec and improved the dependence on A by

$$H = O(A^8 e^{-u} (\log z)^{-1/3}),$$

where the O-constant is an absolute value.

The aim of this paper is to concentrate on the case where u is bounded, to weaken the condition (Ω_2), and to improve the dependence on A.

THEOREM 1. *Let $a \geqslant 2$; then, under the assumption*

(Ω_a): *There exists a constant $A \geqslant 2$, such that*

$$\frac{V(v)}{V(w)} \leqslant \frac{\log w}{\log v}\left(1 + \frac{A}{\log v}\right) \qquad \text{if} \quad a \leqslant v < w,$$

(1.2) *and* (1.3) *hold true with*

$$H = O_a\left(\frac{1}{V(a)} A^{4+\alpha} (\log A)^{2+\alpha} (\log z)^{-\alpha}\right), \qquad 0 < \alpha \leqslant \tfrac{1}{3}. \tag{1.5}$$

If $\log z > KA \log A$ *for some constant K depending at most on a,* (1.2) *and* (1.3) *are even valid with*

$$H = \frac{1}{V(a)} A e^{-u} (\log z)^{-\alpha}, \qquad 0 < \alpha \leqslant \tfrac{1}{3}. \tag{1.6}$$

Remarks. In the first part of the proof we use the fundamental identity of Halberstam and Richert [2, 4]; the second part is, apart from introducing the condition (Ω_a) instead of (Ω_2), closely related to Richert's unpublished manuscript [11].

The constant $\frac{1}{3}$ may be replaced by any constant α_0 for which

$$\frac{e^4}{3^{\alpha_0+1}} \int_3^{\infty} e^{-t} t^{\alpha_0}\, dt < 1, \qquad \text{f.e. } \alpha_0 = 0.36.$$

With the further conditions on $\omega(p)$:

(Ω_1): *There exists a constant* $A_1 \geqslant 2$, *such that* $0 \leqslant \omega(p)/p \leqslant 1 - 1/A_1$,

it follows immediately that $1/V(a) \leqslant A^{a-2}$. Sieving polynomial sequences, the natural condition instead of (Ω_a) is the following (cf. [3]):

$(\Omega_{2,a})$: *There exists a constant* $A_2 \geqslant 2$, *such that*

$$\sum_{v \leqslant p < w} \frac{\omega(p)}{p} \log p \leqslant \log \frac{w}{v} + A_2 \qquad \textit{if} \quad a \leqslant v < w.$$

This condition together with (Ω_1) implies (Ω_a) (cf. [3, Lemma 2.3]).

In order to hold the A_2-dependence on A small we introduce the rather weak condition

$$(\Omega_0): \quad \sum_{v \leqslant p < w} \left(\frac{\omega(p)}{p}\right)^2 \leqslant \frac{A_0}{\log v} \qquad \textit{for some constant } A_0 \geqslant 2, \quad \textit{if} \quad a \leqslant v < w.$$

THEOREM 2. *(Ω_0), (Ω_1), $(\Omega_{2,a})$ imply (Ω_a) with* $A = (A_2 + A_0 A_1)\, e^{(A_2 + A_0 A_1)/\log a}$.

This theorem shows the advantage of replacing (Ω_2) by (Ω_a) for some $a > 2$. Choosing $\log a$ great keeps the A_2-dependence on A and, therefore on H small. A direct consequence of Theorem 1 and 2 is the following:

THEOREM 3. *Let the conditions (Ω_0), (Ω_1), and $(\Omega_{2,2})$ be satisfied. Then* (1.2) *and* (1.3) *are valid with*

$$H = O_{A_0, A_1, \varepsilon}\left(\frac{e^{\varepsilon A_2}}{(\log z)^{1/3}}\right), \qquad \varepsilon > 0.$$

As applications of Theorem 3 to the sieving of polynomial sequences we consider Theorem 5.4 and 5.9 in the book of Halberstam and Richert [3], and make the error terms explicit in the discriminant Δ of the corresponding polynomial.

THEOREM 4. *Let $f(m)$ $(\neq m)$ be an irreducible polynomial of degree n*

and discriminant Δ with integral coefficients and positive leading coefficient. Let $\rho_f(p)$ be the number of incongruent solutions $\bmod p$ *of the congruence $f(m) \equiv 0(p)$ and suppose that $\rho_f(p) < p$ for all p. Then for $1 < y \leqslant x$,*

$$|\{m\colon x-y<m\leqslant x, f(m)=p\}|$$
$$\leqslant 2\prod_p\left(1-\frac{\rho_f(p)-1}{p-1}\right)\frac{y}{\log y}\left(1+O_{n,\varepsilon}\left(\frac{|\Delta|^{\varepsilon\log\log 3|\Delta|}}{(\log y)^{1/3}}\right)\right).$$

THEOREM 5. *Let $f(m)$ and $\rho_f(p)$ be as in Theorem* 3 *and suppose further that*

$$\rho_f(p) < p-1 \qquad \text{if} \quad p \nmid f(0).$$

Let k, l be integers and x a real number, such that with some constant B

$$1 \leqslant k \leqslant (\log x)^B, \qquad (k,l)=1.$$

Then for $x \geqslant 3$,

$$|\{p\colon p\leqslant x, p\equiv l(k), f(p)=p'\}|$$
$$\leqslant 8\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\prod_{\substack{2<p\nmid f(0)\\ p\nmid k}}\left(1-\frac{\rho_f(p)-1}{p-2}\right)$$
$$\times\prod_{\substack{2<p\mid f(0)\\ p\nmid k}}\left(1-\frac{\rho_f(p)-2}{p-2}\right)\prod_{2<p\mid k}\frac{p-1}{p-2}\frac{x}{\Phi(k)\log^2 x}$$
$$\times\left(1+O_{n,\varepsilon,B}\left(\frac{|\Delta|^{\varepsilon\log\log 3|\Delta|}}{(\log x)^{1/3}}\right)\right).$$

In the case $n=1$, the remainder term is independent of Δ.

For the proof of the corresponding theorems in [3], Halberstam and Richert needed the following:

$$\sum_{v\leqslant p<w}\frac{\rho_f(p)}{p}\log p=\log\frac{w}{v}+O_{n,\Delta}(1) \qquad \text{if} \quad 2\leqslant v<w. \tag{1.7}$$

This old result of Nagell [10] implies $(\Omega_{2,2})$ with $A_2 = O_{n,\Delta}(1)$. An explicit result in Δ is given by

THEOREM 6. *Let $f(x)$ and $\rho_f(p)$ be as above. Then for $n \geqslant 2$,*

$$\sum_{v\leqslant p<w}\frac{\rho_f(p)}{p}\log p\leqslant\log\frac{w}{v}+O_n(\log 3|\Delta|\log\log 3|\Delta|) \qquad \text{if } 2\leqslant v<w. \tag{1.8}$$

Since $\rho_f(p) \leqslant n$, (1.8) is even valid for $n=1$ with an absolute O-constant. The proofs of Theorems 4 and 5 are direct consequences of Theorem 3 if in [3] (1.7) is replaced by (1.8).

2. Proof of Theorem 1

We start with some necessary basic notations (cf. [2]). For $n>1$ let $p(n)$ denote the least, and $q(n)$ the largest, prime factor of n. Define $p(1)=\infty$ and $q(1)=1$. Let $\chi(\cdot)$ be a function defined on the set of all positive integer divisors of $\mathscr{P}(z)$ and require that $\chi(1)=1$. Define

$$\bar{\chi}(n)=\begin{cases} 0 & \text{if } \ n=1, \\ \chi\left(\dfrac{n}{p(n)}\right)-\chi(n) & \text{if } \ n>1, n \mid \mathscr{P}(z). \end{cases} \tag{2.1}$$

Lemma 2.1 (Fundamental identity, cf. [2, 4]). *Let $h(\cdot)$ be any arithmetic function. Then*

$$\sum_{d \mid \mathscr{P}(z)} \mu(d)\, h(d) = \sum_{d \mid \mathscr{P}(z)} \mu(d)\, \chi(d)\, h(d) + \sum_{m \mid \mathscr{P}(z)} \mu(m)\, \bar{\chi}(m) \sum_{d \mid \mathscr{P}(p(m))} \mu(d)\, h(md).$$

Proof. Let $d=p_1 p_2 \cdots p_r$, $p_1 > \cdots > p_r$. Then

$$\sum_{\substack{m \mid d \\ q(d/m) < p(m)}} \bar{\chi}(m) = \sum_{i=1}^{r} (\chi(p_1 \cdots p_{i-1}) - \chi(p_1 \cdots p_i)) = 1-\chi(d).$$

Multiplying both sides by $\mu(d)\, h(d)$ and summing over all divisors d of $\mathscr{P}(z)$ gives the desired result.

Suppose now that $\chi^+(\cdot)$ and $\chi^-(\cdot)$ are choices of χ such that

$$\mu(m)\, \bar{\chi}^+(m) \leqslant 0 \qquad \text{and} \qquad \mu(m)\, \bar{\chi}^-(m) \geqslant 0 \qquad \text{for every} \quad m \mid \mathscr{P}(z). \tag{2.2}$$

Then from (1.1) and the fundamental identity with $h(d)=|\mathscr{A}_d|$,

$$\sum_{d \mid \mathscr{P}(z)} \mu(d)\, \chi^-(d)\, |\mathscr{A}_d| \leqslant S(\mathscr{A}, \mathscr{P}, z) \leqslant \sum_{d \mid \mathscr{P}(z)} \mu(d)\, \chi^+(d)\, |\mathscr{A}_d|. \tag{2.3}$$

Now replace $|\mathscr{A}_d|$ by $(\omega(d)/d)X + R_d$ and apply Lemma 2.1 with $h(d)=\omega(d)/d$ in the two sums in (2.3). Then this leads to

$$S(\mathscr{A}, \mathscr{P}, z) \leqslant XV(z) - X \sum_{d \mid \mathscr{P}(z)} \mu(d)\, \bar{\chi}^{+}(d) \frac{\omega(d)}{d} V(p(d)) + \sum_{d \mid \mathscr{P}(z)} \mu(d)\, \chi^{+}(d)\, R_d \tag{2.4}$$

and

$$S(\mathscr{A}, \mathscr{P}, z) \geqslant XV(z) - X \sum_{d \mid \mathscr{P}(z)} \mu(d)\, \bar{\chi}^{-}(d) \frac{\omega(d)}{d} V(p(d)) + \sum_{d \mid \mathscr{P}(z)} \mu(d)\, \chi^{-}(d)\, R_d. \tag{2.5}$$

Next we choose as special weights $\chi^{\pm}$ the Rosser–Iwaniec weights of the linear sieve (cf. [5]) defined as follows: For $1 < n = p_1 \cdots p_r$, $p_1 > \cdots > p_r$, let

$$\chi^{\pm}(n) = \prod_{1 \leqslant l \leqslant r} \eta^{\pm}(p_1 \cdots p_l) \tag{2.6}$$

so that by (2.1),

$$\bar{\chi}^{\pm}(n) = \chi^{\pm}\left(\frac{n}{p(n)}\right)(1 - \eta^{\pm}(n)), \tag{2.7}$$

where

$$\eta^{+}(m) = \begin{cases} 1 & \text{if } \mu(m) = 1 \quad \text{or if } \mu(m) = -1 \text{ and } p^2(m)m < y, \\ 0 & \text{otherwise;} \end{cases} \tag{2.8}$$

and

$$\eta^{-}(m) = \begin{cases} 1 & \text{if } \mu(m) = -1 \quad \text{or if } \mu(m) = 1 \text{ and } p^2(m) < y, \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

Obviously these weights satisfy (2.2). Note that $\chi^{\pm}(n) = 1$ implies that $n < y$. Inserting (2.6)–(2.9) in (2.4) and (2.5) leads to

$$(-1)^v \{S(\mathscr{A}, \mathscr{P}, z) - XV(z)\} \leqslant X \sum_{r=1}^{\infty} S_{r,z}^{(-)^v}(u) + \sum_{\substack{d < y \\ d \mid \mathscr{P}(z)}} |R_d|, \qquad v = 1, 2 \tag{2.10}$$

where

$$S_{r,z}^{+}(u)=\sum_{\substack{p_{2r-1}<\cdots<p_1<z\\ p_{2j-1}^3 p_{2j-2}\cdots p_1<y\, j=1,\dots,r-1\\ p_{2r-1}^3 p_{2r-2}\cdots p_1\geqslant y}} \frac{\omega(p_1\cdots p_{2r-1})}{p_1\cdots p_{2r-1}} V(p_{2r-1}) \tag{2.11}$$

and

$$S_{r,z}^{-}(u)=\sum_{\substack{p_{2r}<\cdots<p_1<z\\ p_{2j}^3 p_{2j-1}\cdots p_1<y,\, j=1,\dots,r-1\\ p_{2r}^3 p_{2r-1}\cdots p_1\geqslant y}} \frac{\omega(p_1\cdots p_{2r})}{p_1\cdots p_{2r}} V(p_{2r}). \tag{2.12}$$

The summation condition in (2.11) and (2.12) imply that (cf. (1.4))

$$S_{r,z}^{\pm}(u)=0 \qquad \text{for} \quad u\geqslant 2r+2-\frac{1\pm 1}{2}, \qquad r\geqslant 1. \tag{2.13}$$

Define as in [5]

$$T_{R,z}^{\pm}(u)=\sum_{r=1}^{R} S_{r,z}^{\pm}(u) \qquad \text{for} \quad u\geqslant 2-\frac{1\pm 1}{2}, \qquad R\geqslant 1. \tag{2.14}$$

Then we have the following recurrence formula

$$T_{1,z}^{+}(u)=\begin{cases} V(y^{1/3})-V(z), & 1\leqslant u\leqslant 3,\\ 0, & u\geqslant 3,\end{cases} \tag{2.15}$$

$$T_{R,z}^{+}(u)=T_{1,z}^{+}(u)+T_{R,z}^{+}(3), \qquad 1\leqslant u\leqslant 3,\ R\geqslant 1, \tag{2.16}$$

$$T_{R+(1\pm 1)/2,z}^{\pm}(u)=\sum_{y^{1/(2R+2+(1\pm 1)/2)}\leqslant p<z} \frac{\omega(p)}{p}\, T_{R,p}^{\mp}\left(\frac{\log(y/p)}{\log p}\right),$$

$$u\geqslant 2+\frac{1\pm 1}{2},\ R\geqslant 1. \tag{2.17}$$

We try to approximate $S_{r,z}^{\pm}(u)$ and $T_{R,z}^{\pm}(u)$ by functions $\phi_r^{\pm}(u)$ and $\Phi_R^{\pm}(u)$, respectively. Define

$$\phi_1^{+}(u)=\begin{cases} \dfrac{3}{u}-1, & 1\leqslant u\leqslant 3,\\ 0, & u\geqslant 3,\end{cases}$$

$$\phi_r^{+}(u)=\frac{3}{u}\,\phi_r^{+}(3), \qquad 1\leqslant u\leqslant 3,\ r\geqslant 2,$$

$$\phi_{r+(1\pm 1)/2}^{\pm}(u)=\frac{1}{u}\int_u^{\infty} \phi_r^{\mp}(t-1)\,dt, \qquad u\geqslant 2+\frac{1\pm 1}{2},\ r\geqslant 1, \tag{2.18}$$

and

$$\Phi_R^{\pm}(u)=\sum_{r=1}^{R}\phi_r^{\pm}(u), \qquad u\geqslant 2-\frac{1\pm 1}{2},\ R\geqslant 1.$$

Note that the functions $\phi_r^{\pm}(u)$ are non-negative and continuous and that

$$\begin{aligned}
&\phi_r^{\pm}(u)=0 && \text{for}\quad u\geqslant 2r+2-\frac{1\pm 1}{2},\\
&\Phi_1^{+}(u)=\begin{cases}\dfrac{3}{u}-1, & 1\leqslant u\leqslant 3,\\ 0 & u\geqslant 3,\end{cases}
\end{aligned}\tag{2.19}$$

$$\Phi_R^{+}(u)=\frac{3}{u}-1+\frac{3}{u}\,\Phi_R^{+}(3), \qquad 1\leqslant u\leqslant 3,\ R\geqslant 1,\tag{2.20}$$

$$\Phi_{R+(1\pm 1)/2}^{\pm}(u)=\frac{1}{u}\int_u^{\infty}\Phi_R^{\mp}(t-1)\,dt, \qquad u\geqslant 2+\frac{1\pm 1}{2},\ R\geqslant 1.\tag{2.21}$$

Next we shall prove by induction that

$$\Phi_R^{\pm}(u)\leqslant 18e^{-u}, \qquad u\geqslant 2-\frac{1\pm 1}{2},\ R\geqslant 1,\tag{$2.22_R^{\pm}$}$$

$$\Phi_R^{+}(3)\leqslant 6e^{-2}, \qquad R\geqslant 1.\tag{2.23_R}$$

Proof. Both (2.22_1^{+}) and (2.23_1) are direct consequences of (2.19). By (2.21), (2.22_R^{+}) yields

$$\Phi_R^{-}(u)\leqslant\frac{18}{u}\int_u^{\infty}e^{1-t}\,dt\leqslant 18e^{-u} \qquad \text{for}\quad u\geqslant 4.$$

(2.23_R) implies by (2.20),

$$\Phi_R^{+}(u)\leqslant\frac{3}{u}-1+\frac{18}{u}e^{-2} \qquad \text{for } 1\leqslant u\leqslant 3.$$

Hence by (2.21) it follows for $2\leqslant u\leqslant 4$ that

$$\Phi_R^{-}(u)\leqslant\frac{18}{u}\int_4^{\infty}e^{1-t}\,dt+\frac{1}{u}\int_u^{4}\left(\frac{3+18e^{-2}}{t-1}-1\right)dt<18e^{-u},$$

i.e., (2.22_R^{-}) for $2\leqslant u\leqslant 4$.

Next we assume (2.22_R^{-}). Then by (2.21),

$$\Phi_{R+1}^{+}(u)\leqslant\frac{18}{u}\int_u^{\infty}e^{1-t}\,dt=\frac{18e^{1-u}}{u}.$$

For $u \geqslant 3$ this implies (2.22^{+}_{R+1}), and, for $u=3$, (2.23_{R+1}). By (2.20) and (2.23_{R+1}),

$$\Phi^{+}_{R+1}(u) \leqslant \frac{3}{u} - 1 + \frac{18}{u} e^{-2} < 18 e^{-u} \qquad \text{for} \quad 1 \leqslant u \leqslant 3,$$

i.e., (2.22^{+}_{R+1}) for $1 \leqslant u \leqslant 3$. Q.E.D.

The functions $\Phi^{\pm}_{R}(u)$ are non-negative and non-decreasing in R so that (2.22) proves the existence of

$$\Phi^{\pm}(u) = \lim_{R \to \infty} \Phi^{\pm}_{R}(u), \qquad u \geqslant 2 - (1 \pm 1)/2. \tag{2.24}$$

By (2.20) and (2.22) we have

$$\Phi^{\pm}_{R}(u) \leqslant \Phi^{\pm}(u) \leqslant 18 e^{-u}, \qquad u \geqslant 2 - (1 \pm 1)/2,\ R \geqslant 1, \tag{2.25}$$

$$\Phi^{+}(u) = \frac{3}{u} - 1 + \frac{3}{u} \Phi^{+}(3), \ \ 1 \leqslant u \leqslant 3; \tag{2.26}$$

and by (2.18) and uniform convergence it follows from (2.21) that

$$\Phi^{\pm}(u) = \frac{1}{u} \int_{u}^{\infty} \Phi^{\mp}(t-1)\, dt, \qquad u \geqslant 2 + \frac{1 \pm 1}{2}. \tag{2.27}$$

Now we are in a position to estimate $T^{\pm}_{R,z}(u)$. First of all, by (2.11), (2.13), and (2.14), since $V(z) \leqslant 1$, we have for any $\xi \geqslant 1$,

$$\begin{aligned} T^{+}_{R,z}(u) &\leqslant \sum_{\substack{r=1 \\ 2r-1>u-2}}^{R} \ \sum_{p_{2r-1} < \cdots < p_1 < z} \frac{\omega(p_1 \cdots p_{2r-1})}{p_1 \cdots p_{2r-1}} \\ &\leqslant \sum_{\substack{r=1 \\ 2r-1>u-2}}^{R} \frac{1}{(2r-1)!} \left(\sum_{p<z} \frac{\omega(p)}{p} \right)^{2r-1} \\ &\leqslant \xi^{2-u} \sum_{n>u-2} \frac{1}{n!} \left(\xi \sum_{p<z} \frac{\omega(p)}{p} \right)^{n}. \end{aligned}$$

Since $S^{-}_{r,z}(u) = 0$ for $u \geqslant 2r+2$ we obtain the same estimate for $T^{-}_{R,z}(u)$. This implies for any $\xi \geqslant 1$ and $\alpha > 0$,

$$\begin{aligned} T^{\pm}_{R,z}(u) \leqslant \frac{V(z)}{V(a)} (\log z)^{-\alpha} \exp \Bigg(\xi \sum_{p<z} \frac{\omega(p)}{p} + \log \frac{V(a)}{V(z)} \\ + \alpha \log \log z + (2-u) \log \xi \Bigg). \end{aligned} \tag{2.28}$$

By choosing $\xi = e$ we obtain for $z \leqslant a$,

$$T^{\pm}_{R,z}(u) \ll_a \frac{V(z)}{V(a)} (\log z)^{-\alpha} e^{-u}. \tag{2.29}$$

In the other case, using $x < \log(1/(1-x))$ for $0 < x < 1$, we find that

$$\sum_{a \leqslant p < z} \frac{\omega(p)}{p} < \sum_{a \leqslant p < z} \log\left(\frac{1}{1-(\omega(p)/p))}\right) = \log \frac{V(a)}{V(z)}.$$

Inserting this in (2.28) and applying (Ω_a) gives for $z > a$

$$\begin{aligned} T^{\pm}_{R,z}(u) \leqslant \frac{V(z)}{V(a)} (\log z)^{-a} \exp((\xi + 1 + \alpha) \log\log z \\ + (\xi + 1) \log 3A + (2-u) \log \xi). \end{aligned} \tag{2.30}$$

Put

$$\log z_0 = KA \log A \tag{2.31}$$

with sufficiently large K. Then (2.30) implies for $\log a < \log z \leqslant \log z_0$, by choosing $\xi = 1$,

$$T^{\pm}_{R,z}(u) \ll_K \frac{V(z)}{V(a)} (\log z)^{-\alpha} A^{4+\alpha} (\log A)^{2+\alpha} \tag{2.32}$$

and for $\log z \geqslant \log z_0$, $u \geqslant 2(e^2+2) \log\log z$, by choosing $\xi = e^2$ and $\alpha = 1$,

$$T^{\pm}_{R,z}(u) \leqslant \frac{V(z)}{V(a)} (\log z)^{-1} e^{-u} \frac{1}{K}. \tag{2.33}$$

In the remaining cases, i.e. (cf. (2.31)),

$$\log z \geqslant KA \log A \qquad \text{and} \qquad u \leqslant u_1 := 2(e^2+2) \log\log z, \qquad K \text{ sufficiently large}, \tag{2.34}$$

we prove by induction (cf. (2.24))

$$T^{\pm}_{R,z}(u) \leqslant V(z) \left\{ \Phi^{\pm}(u) + A^* \frac{e^{-u}}{(\log z)} \alpha \right\}, \qquad A^* = \frac{A}{V(a)}. \tag{2.35$^{\pm}_{R}$}$$

As a consequence of (Ω_a) we need for the proof of (2.35)

LEMMA 2.2. *Let $a \leqslant z_1 \leqslant z$ and $B(x)$ be a positive, continuous, and increasing function in the interval $z_1 \leqslant p < z$. Then under* (Ω_α)

$$\sum_{z_1 \leqslant p < z} \frac{\omega(p)}{p} \frac{V(p)}{V(z)} \frac{\log p}{\log z} B(p) \leqslant \int_{z_1}^{z} \frac{B(x)}{x \log x} dx + \frac{2AB(z)}{\log z_1}.$$

This is Lemma 21 in [5]. In order to apply (Ω_a) and Lemma 2.2 we choose $K = K(a)$ such that

$$u_1 \geqslant 4 \qquad \text{and} \qquad \frac{1}{u_1} \log y \geqslant \frac{1}{u_1} \log z \geqslant \log a. \tag{2.36}$$

By (2.15), $T^{+}_{1,z}(u) = 0$ for $u \geqslant 3$ and for $1 \leqslant u \leqslant 3$ we obtain by (Ω_a) and (2.20)

$$\begin{aligned} T^{+}_{1,z}(u) &\leqslant V(z)\left(\frac{3}{u} - 1 + \frac{9A}{\log z}\right) \\ &\leqslant V(z)\left(\Phi^{+}(u) + \frac{9Ae^3}{\log z} e^{-u}\right), \end{aligned} \tag{2.37}$$

which proves (2.35^{+}_{1}) by (2.34) for K large enough.

Now assume $(2.35^{\pm}_{R})$ and put

$$z_1 = y^{1/u_1}.$$

Replace y in $(2.35^{\pm}_{R})$ by y/p and z by p. Then, by (2.17) for $u \geqslant 2 + (1 \pm 1)/2$,

$$\begin{aligned} &T^{\pm}_{R+(1\pm 1)/2,z}(u) \\ &\quad\leqslant T^{\pm}_{R+(1\pm 1)/2,z_1}(u_1) + V(z) \log z \sum_{z_1 \leqslant p < z} \frac{\omega(p)}{p} \frac{V(p)}{V(z)} \frac{\log p}{\log z} \\ &\quad\times \left\{ \Phi^{\mp}\left(\frac{\log y}{\log p} - 1\right) \Big/ \log p + A^* \frac{e^{1 - \log y/\log p}}{(\log p)^{1+\alpha}} \right\}. \end{aligned} \tag{2.38}$$

By (2.27), $\Phi^{\mp}(\log y/\log p - 1)$ is increasing in p and so is $e^{1-\log y/\log p}/(\log p)^{1+\alpha}$. Since, by (2.36), $z_1 \geqslant a$, Lemma 2.2 is applicable and leads to

$$T^{\pm}_{R+(1\pm 1)/2,z}(u)$$

$$\leqslant T^{\pm}_{R+(1\pm 1)/2,z_1}(u_1)+V(z)$$

$$\times \log z\left[\int_{z_1}^{z}\left(\frac{\Phi^{\mp}(\log y/\log x-1)}{x\log^2 x}+A^*\frac{e^{1-\log y/\log x}}{x\log x)^{2+\alpha}}\right)dx\right.$$

$$\left.+\frac{2Au_1}{\log y}\left(\frac{\Phi^{\mp}(u-1)}{\log z}+A^*\frac{e^{1-u}}{(\log z)^{1+\alpha}}\right)\right].$$

Estimating $T^{\pm}_{R+(1\pm 1)/2,z_1}(u_1)$ by (2.33) and $\Phi^{\mp}(u-1)$ by (2.25) and substituting in the integral $t=\log y/\log x$ gives, in view of (2.34),

$$T^{\pm}_{R+(1\pm 1)/2,z}(u)$$

$$\leqslant V(z)\frac{\log z}{\log y}\int_u^{u_1}\Phi^{\mp}(t-1)\,dt$$

$$+\frac{V(z)}{\log^\alpha z}A^*e^{-u}\left\{\frac{e^{1+u}}{u^{1+\alpha}}\int_u^\infty e^{-t}t^\alpha\,dt+O\left(\frac{\log K}{K}\right)+O\left(\frac{\log KA}{(KA)^{1-\alpha}}\right)\right\},\tag{2.39}$$

where the O-constants are absolute and in particular independent of A and K. By partial integration

$$\frac{e^{1+u}}{u^{1+\alpha}}\int_u^\infty e^{-t}t^\alpha\,dt\leqslant\frac{e}{u}\left(1+\frac{\alpha}{u}+\frac{\alpha(\alpha-1)}{u^2}+\frac{\alpha(\alpha-1)(\alpha-2)}{u^3}\right)$$

$$\leqslant 1-3\cdot 10^{-3}\qquad\text{for}\quad u\geqslant 3,\ \alpha\leqslant\tfrac{1}{3}.$$

Hence, for sufficiently large K, (2.39) gives in view of (2.27),

$$T^{\pm}_{R+(1\pm 1)/2,z}(u)\leqslant V(z)\,\Phi^{\pm}(u)+(1-10^{-3})\frac{V(z)}{\log^\alpha z}A^*e^{-u}\qquad\text{for } u\geqslant 3,\tag{2.40}$$

i.e. (2.35^+_R) for $u\geqslant 1$ implies (2.35^-_R) for $u\geqslant 3$ and (2.35^-_R) for $u\geqslant 2$ implies (2.35^+_{R+1}) for $u\geqslant 3$.

Next we prove that (2.35^+_R) for $u\geqslant 1$ also implies (2.35^-_R) for $2\leqslant u\leqslant 3$. By (2.16), (2.37), and (2.35^+_R) for $u=3$

$$T^+_{R,z}(u)\leqslant V(z)\left\{\frac{3}{u}-1+\frac{9A}{\log z}+\Phi^+(3)+\frac{A^*e^{-3}}{\log^\alpha z}\right\}\qquad\text{for}\quad 1\leqslant u\leqslant 3,$$

i.e., by (2.26)

$$T^{+}_{R,p}\left(\frac{\log y}{\log p}-1\right)\leqslant V(z)\left\{\Phi^{+}\left(\frac{\log y}{\log p}-1\right)+\frac{9A}{\log p}+\frac{A^{*}e^{-3}}{\log^{\alpha} p}\right\}$$
$$\text{for}\quad 2\leqslant\frac{\log y}{\log p}\leqslant 4.$$

Now we return to (2.38) and use this by splitting up the sum there. Thus for $2\leqslant u\leqslant 4$,

$$\begin{aligned}
&T^{-}_{R,z}(u)\\
&\quad\leqslant T^{-}_{R,z_1}(u_1)+V(z)\log z\sum_{z_1\leqslant p<z}\frac{\omega(p)\,V(p)}{pV(z)}\frac{\log p}{\log z}\frac{\Phi^{+}(\log y/\log p-1)}{\log p}\\
&\quad\quad+V(y^{1/4})(\log^{1/4}y)\sum_{z_1\leqslant p<y^{1/4}}\frac{\omega(p)}{p}\frac{V(p)}{V(y^{1/4})}\frac{\log p}{\log^{1/4}y}A^{*}\frac{e^{1-\log y/\log p}}{(\log p)^{1+\alpha}}\\
&\quad\quad+V(z)\sum_{y^{1/4}\leqslant p<z}\frac{\omega(p)}{p}\frac{V(p)}{V(z)}\left(\frac{9A}{\log p}+\frac{A^{*}e^{-3}}{\log^{\alpha}p}\right).
\end{aligned}\tag{2.41}$$

The first two terms on the right of (2.41) are as in (2.39):

$$\leqslant V(z)\,\Phi^{-}(u)+\frac{V(z)}{\log^{\alpha}z}A^{*}e^{-u}\left\{O\left(\frac{\log K}{K}\right)+O\left(\frac{\log KA}{(KA)^{1-\alpha}}\right)\right\}\tag{2.42}$$

and similarly the next term is, by Lemma 2.2, (Ω_a) and (2.34)

$$\begin{aligned}
&\leqslant\frac{V(z)\log z}{(\log y)^{1+\alpha}}\left(1+\frac{4A}{\log y}\right)A^{*}\left\{e\int_4^{\infty}e^{-t}t^{\alpha}\,dt+O\left(\frac{\log K}{K}\right)\right\}\\
&\leqslant\frac{V(z)}{\log^{\alpha}z}A^{*}e^{-u}\left\{\left(1+O\left(\frac{\log K}{K}\right)\right)\frac{e^{1+u}}{u^{1+\alpha}}\int_4^{\infty}e^{-t}t^{\alpha}\,dt+O\left(\frac{\log K}{K}\right)\right\}.
\end{aligned}\tag{2.43}$$

The application of (Ω_a) and Lemma 2.2 is justified, since by (2.36) $y^{1/4}\geqslant y^{1/u_1}\geqslant a$. The last term in (2.41) is estimated trivially, using Buchstab's identity [3, Lemma 7.1]

$$\sum_{y^{1/4}\leqslant p<z}\frac{\omega(p)}{p}\frac{V(p)}{V(z)}=\frac{V(y^{1/4})}{V(z)}-1$$

and (Ω_a), by

$$\leqslant V(z)\left\{\frac{36A}{\log y}+A^*\frac{e^{-3}4^\alpha}{\log^\alpha y}\right\}\left\{\frac{4}{u}\left(1+\frac{4A}{\log y}\right)-1\right\}$$

$$\leqslant\frac{V(z)}{\log^\alpha z}A^*e^{-u}\left\{\frac{e^{u-3}4^\alpha}{u^\alpha}\left(\frac{4}{u}-1\right)\right.$$

$$\left.\times\left(1+O\left(\frac{\log K}{K}\right)\right)+O((KA)^{-1+\alpha})\right\}. \tag{2.44}$$

By partial integration

$$\frac{e^{1+u}}{u^{1+\alpha}}\int_4^\infty e^{-t}t^\alpha\,dt+e^{u-3}\left(\frac{4}{u}\right)^\alpha\left(\frac{4}{u}-1\right)$$

$$\leqslant\frac{e^{u-3}}{u}\left(\frac{4}{u}\right)^\alpha\left(5+\frac{\alpha}{4}-u\right)\leqslant\frac{4}{5}\qquad\text{for}\quad 2\leqslant u\leqslant 4.$$

This combined with (2.41), (2.42), (2.43), and (2.44) proves (2.35_R^-) for $2\leqslant u\leqslant 3$ by choosing K large enough.

It remains to prove (2.35_{R+1}^+) for $1\leqslant u\leqslant 3$. By (2.16), (2.37), and (2.40) for $u=3$ and (2.26)

$$T_{R+1,z}^+(u)=T_{1,z}^+(u)+T_{R+1,z}^+(3)$$

$$\leqslant V(z)\left(\frac{3}{u}-1+\Phi^+(3)+\frac{9A}{\log z}\right)+A^*(1-10^{-3})\frac{V(z)\,e^{-3}}{\log^\alpha z}$$

$$\leqslant V(z)\,\Phi^+(u)+A^*\frac{V(z)\,e^{-u}}{\log^\alpha z}\left((1-10^{-3})\,e^{u-3}+\frac{9e^u}{(\log z)^{1-\alpha}}\right).$$

This proves (2.35_{R+1}^+) for $1\leqslant u\leqslant 3$.

By (2.10) and (2.14) our results (2.29) (2.32), (2.33), and (2.35) imply that

$$\begin{aligned} S(\mathscr{A},\mathscr{P},z)&\leqslant XV(z)(1+\Phi^+(u)+H)+\sum_{\substack{d<y\\ d\mid\mathscr{P}(z)}}|R_d|\\ S(\mathscr{A},\mathscr{P},z)&\geqslant XV(z)(1-\Phi^-(u)-H)-\sum_{\substack{d<y\\ d\mid\mathscr{P}(z)}}|R_d|\end{aligned} \tag{2.45}$$

for some H satisfying (1.5) and (1.6).

It remains to deal with the functions $\Phi^\pm(u)$. Put

$$\sigma(u)=u(\Phi^+(u)+\Phi^-(u))\qquad\text{for}\quad u\geqslant 2. \tag{2.46}$$

By (2.26)

$$\Phi^+(u) = C/u - 1 \qquad \text{for} \quad 1 \leqslant u \leqslant 3 \tag{2.47}$$

and by (2.27)

$$\begin{aligned} u\Phi^-(u) &= 2\Phi^-(2) - \int_2^u \Phi^+(t-1)\,dt \\ &= -C\log(u-1) + u - 2, \qquad 2 \leqslant u \leqslant 4. \end{aligned} \tag{2.48}$$

Thus by (2.46)

$$\sigma(u) = 2\Phi^-(2) - 2 + C - C\log(u-1), \qquad 2 \leqslant u \leqslant 3 \tag{2.49}$$

and by (2.27)

$$\sigma'(u) = -\frac{1}{u-1}\,\sigma(u-1), \qquad u \geqslant 3,$$

so that in view of (2.25)

$$(u-1)\,\sigma(u) = \int_{u-1}^u \sigma(t)\,dt, \qquad u \geqslant 3.$$

and with partial integration this yields by (2.49)

$$\int_2^3 (t-1)\,\sigma'(t)\,dt = (t-1)\,\sigma(t)\Big|_2^3 - \int_2^3 \sigma(t)\,dt = 2 - 2\Phi^-(2) - C. \tag{2.50}$$

On the other hand, by (2.49)

$$\int_2^3 (t-1)\,\sigma'(t)\,dt = -C. \tag{2.51}$$

Comparing (2.50) and (2.51) implies that $\Phi^-(2) = 1$, so that we may put

$$\Phi^-(u) = 1 \qquad \text{for} \quad 1 \leqslant u \leqslant 2. \tag{2.52}$$

Next we consider

$$w(u) = (1/C)(2 + \Phi^+(u) - \Phi^-(u)), \qquad u \geqslant 1.$$

By (2.27), (2.47), (2.48), and (2.52)

$$\begin{aligned} (uw(u))' &= w(u-1), \qquad && u \geqslant 2 \\ w(u) &= 1/u, && 1 \leqslant u \leqslant 2. \end{aligned}$$

De Bruijn proved for this function (cf. [3, p. 225])

$$w(\infty) = e^{-\gamma}.$$

Since by (2.25) $w(\infty) = 2/C$, it now follows that

$$C = 2e^{\gamma}. \tag{2.53}$$

Define

$$F(u) = 1 + \Phi^{+}(u), \qquad f(u) = 1 - \Phi^{-}(u), \qquad u \geqslant 1. \tag{2.54}$$

Then by (2.47), (2.52), and (2.53)

$$F(u) = 2e^{\gamma}/u, \qquad f(u) = 0, \qquad 1 \leqslant u \leqslant 2, \tag{2.55}$$

and by (2.47), (2.52), and (2.27)

$$(uF(u))' = f(u-1), \qquad (uf(u))' = F(u-1), \qquad u \geqslant 2. \tag{2.56}$$

Thus $F(u)$ and $f(u)$ are the well-known functions of the linear sieve. Theorem 1 is now a direct consequence of (2.45), (2.54), (2.55), and (2.56).

3. Proof of Theorem 2

By $(\Omega_{2,a})$ we have (cf. [3, Lemma 2.3])

$$\sum_{v \leqslant p < w} \frac{\omega(p)}{p} - \log \frac{\log w}{\log v} \leqslant \frac{A_2}{\log v} \qquad \text{for} \quad a \leqslant v < w.$$

This together with (Ω_0) and (Ω_1) gives for $a \leqslant v < w$,

$$\begin{aligned}
&\frac{V(v)}{V(w)} \frac{\log v}{\log w} \\
&\quad = \exp\left(\sum_{v \leqslant p < w} \frac{\omega(p)}{p} - \log \frac{\log w}{\log v} + \sum_{v \leqslant p < w} \sum_{n=2}^{\infty} \left(\frac{\omega(p)}{p} \right)^{n} \right) \\
&\quad \leqslant \exp\left(\frac{A_2 + A_0 A_1}{\log v} \right) \\
&\quad \leqslant 1 + \frac{A_2 + A_0 A_1}{\log v} \exp\left(\frac{A_2 + A_0 A_1}{\log a} \right),
\end{aligned}$$

which proves Theorem 2.

4. Proof of Theorem 6

First of all we need some basic definitions of algebraic number theory (cf. [9]). Let $g(x)=x^n+a_{n-1}x^{n-1}+\cdots+a_0$, $n\geqslant 2$, be an irreducible polynomial of degree n and discriminant Δ with integral coefficients and let θ be a root of g, i.e., $g(\theta)=0$. By $K=K(\theta)$ we denote the algebraic number field over $\mathbb{Q}$ of degree n induced by θ, i.e., the smallest number field containing both $\mathbb{Q}$ and θ. Let D be the discriminant of $K(\theta)$. Then $\Delta=D$, if $1, \theta, \ldots, \theta^{n-1}$ form an integral basis of $K(\theta)$. Ideals in $K(\theta)$ are denoted by $\mathfrak{A}$ and their norm by $N\mathfrak{A}$; by $\mathscr{P}$ and $N\mathscr{P}$ we always denote prime ideals and norms of prime ideals, respectively. The Dedekind zeta-function $\zeta_K(s)$, $s=\sigma+it$ of $K(\theta)$ is defined for $\sigma>1$ by

$$\zeta_K(s)=\sum_{\mathfrak{A}}\frac{1}{N\mathfrak{A}^s}=\sum_{m=1}^{\infty}\frac{F(m)}{m^s},$$

where $\mathfrak{A}$ runs through all ideals of K and where $F(m)$ is the number of ideals in K with Norm m. $\zeta_K(s)$ can be continued analytically over the whole plane as a meromorphic function; its only pole being a simple pole at $s=1$.

For the proof of (1.7) Nagell [10] started with the well-known formula of Landau [8],

$$\sum_{N\mathscr{P}\leqslant x}\frac{\log N\mathscr{P}}{N\mathscr{P}}=\log x+O(1). \tag{4.1}$$

Since the O-constant may depend on the degree n and the discriminant D, we make at first the right-hand side of (4.1) explicit in D. As in the rational number field let

$$\vartheta(x)=\sum_{N\mathscr{P}<x}\log N\mathscr{P}$$

and

$$\psi(x)=\sum_{l}\vartheta(x^{1/l}).$$

By partial summation it follows immediately that

$$\sum_{v\leqslant N\mathscr{P}<w}\frac{\log N\mathscr{P}}{N\mathscr{P}}=\log\frac{w}{v}+\frac{\vartheta(w)}{w}-\frac{\vartheta(v)}{v}+\int_v^w\frac{\vartheta(t)-t}{t^2}\,dt.$$

Since we are only interested in an upper bound and since $\vartheta(x)\leqslant\psi(x)$ we can use the following version of the prime ideal theorem of Wiertelak:

LEMMA 4.1 [12, Lemma 9]. *Let $E_1 = 1$, if there exists a real simple zero β_1 of $\zeta_K(s)$ in the region*

$$\sigma > 1 - \frac{c_1}{\log(|D|(|t|+2)^n)}, \qquad -\infty < t < \infty,$$

c_1 independent of n and D, and $E_1 = 0$, otherwise. Then there exists a numerical constant $c_2 > 0$ such that

$$\psi(x) = x - E_1 \frac{x^{\beta_1}}{\beta_1} + O\left(x \log 2|D| \exp\left(-\frac{c_2 \log x}{\max(\sqrt{n \log x}, \log 2|D|)}\right)\right).$$

Lemma 4.1 implies that

$$\vartheta(t) - t \leqslant ct \log 2|D| \exp\left(-\frac{c_2 \log t}{\max(\sqrt{n \log t}, \log 2|D|)}\right).$$

Thus,

$$\sum_{v \leqslant N\mathscr{P} < w} \frac{\log N\mathscr{P}}{N\mathscr{P}} \leqslant \log \frac{w}{v} + O_n(\log 2|D|),$$
$$\text{if} \quad \log w > \log v \geqslant \log 3|D| \log\log 3|D|. \quad (4.2)$$

Now we are in a position to proceed like Nagell. Let v_p^f be the number of distinct prime ideals of degree f dividing the rational prime p, then obviously

$$\sum_{v \leqslant N\mathscr{P} < w} \frac{\log N\mathscr{P}}{N\mathscr{P}} = \sum_{v \leqslant p < w} v_p^1 \frac{\log p}{p} + \sum_{f=2}^{n} \sum_{v \leqslant p^f < w} f v_p^f \frac{\log p}{p^f}.$$

The last sum may be estimated by $O_n(1)$. For v_p^1 Dedekind proved in [1] that

$$v_p^1 = \rho_g(p) \qquad \text{if} \quad p \nmid \Delta/D,$$

for the definition of $\rho_g(p)$ (cf. Theorem 4). Thus,

$$\sum_{v \leqslant p < w} \rho_g(p) \frac{\log p}{p}$$
$$\leqslant \sum_{v \leqslant N\mathscr{P} < w} \frac{\log N\mathscr{P}}{N\mathscr{P}} + \sum_{\substack{v \leqslant p < w \\ p \mid \Delta/D}} \rho_g(p) \frac{\log p}{p} + O_n(1). \quad (4.3)$$

Since $\rho_g(p) \leqslant n$ and

$$\sum_{\substack{p \geqslant v \\ p \mid \Delta/D}} \frac{\log p}{p} \ll \log\log 3|\Delta|,$$

(4.2) together with (4.3) imply that

$$\sum_{v \leqslant p < w} \frac{\rho_g(p)}{p} \log p \leqslant \log \frac{w}{v} + O_n(\log 2|\Delta|),$$

$$\text{if} \quad \log v \geqslant \log 3|\Delta| \log\log 3|\Delta|. \quad (4.4)$$

If $\log w \leqslant \log 3|\Delta| \log\log 3|\Delta|$, we estimate the sum in (4.4) trivially by

$$\sum_{v \leqslant p < w} \frac{\rho_g(p)}{p} \log p \leqslant n \sum_{v \leqslant p < w} \frac{\log p}{p} \ll_n \log 3|\Delta| \log\log 3|\Delta|. \quad (4.5)$$

Theorem 6 is now a direct consequence of (4.4) and (4.5) if the coefficient a_n of the polynomial is equal to 1. For the argument in the other case cf. [10].

Acknowledgments

I am very grateful to Professors Halberstam and Richert for showing me their unpublished manuscripts.

References

1. R. Dedekind, Über den Zusammenhang zwischen der Theorie der Ideale und der Theorie der höheren Kongruenzen, *Abh. K. Gesellschaft Wiss. Göttingen* **23** (1878), 1–23.
2. H. Halberstam, Lectures on the linear sieve, University of Texas at Austin, 1982.
3. H. Halberstam and H.-E. Richert, "Sieve Methods," Academic Press, London, 1974.
4. H. Halberstam and H.-E. Richert, A weighted sieve of Greaves type I, *in* "Proceedings Banach Center," Warsaw, 1984, to appear.
5. H. Iwaniec, Rosser's sieve, *Acta Arith.* **36** (1980), 171–202.
6. H. Iwaniec, A new form of the error term in the linear sieve, *Acta Arith.* **37** (1980), 307–320.
7. W. B. Jurkat and H.-E. Richert, An improvement of Selberg's sieve method I, *Acta Arith.* **11** (1965), 217–240.
8. E. Landau, Über die zu einem algebraischen Zahlkörper gehörige Zetafunktion und die Ausdehnung der Tschebyscheff'schen Primzahltheorie auf das Problem der Verteilung der Primideale, *J. Reine Angew. Math.* **125** (1903), 64–188.
9. E. Landau, "Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale," 2nd ed. Chelsea, New York, 1949.
10. T. Nagell, Généralisation d'un théorème de Tschebyscheff, *J. Math. Pures Appl. (8)* **4** (1921), 343–356.
11. H.-E. Richert, The linear sieve, unpublished.
12. K. Wiertelak, On the density of some set of primes, II, *Acta Arith.* **34** (1980), 197–210.